E INVERSE FUNCTIONS

The operations of $+$ and $-$, and $\times$ and $\div$, are **inverse operations** since one ‘undoes’ the other.

In the same way, some functions have **inverse functions** which ‘undo’ each other.

For a function f which converts x to $f(x)$, the **inverse function** f^{-1} converts $f(x)$ back to x.

The inverse function satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

For a given function f, an inverse function f^{-1} only exists if, for any value of $f(x)$, there is only *one* corresponding value of x.

From the **Investigation** you should have found that:

- $f^{-1}(x)$ can be found algebraically by interchanging x and y and then making y the subject of the resulting formula. The new y is $f^{-1}(x)$.
- $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.

Example 8

Self Tutor

Consider $f(x) = \frac{1}{2}x - 1$.

a Find $f^{-1}(x)$.

b Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

c Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

a $y = \frac{1}{2}x - 1$ has inverse function $x = \frac{1}{2}y - 1$ {interchanging x and y}

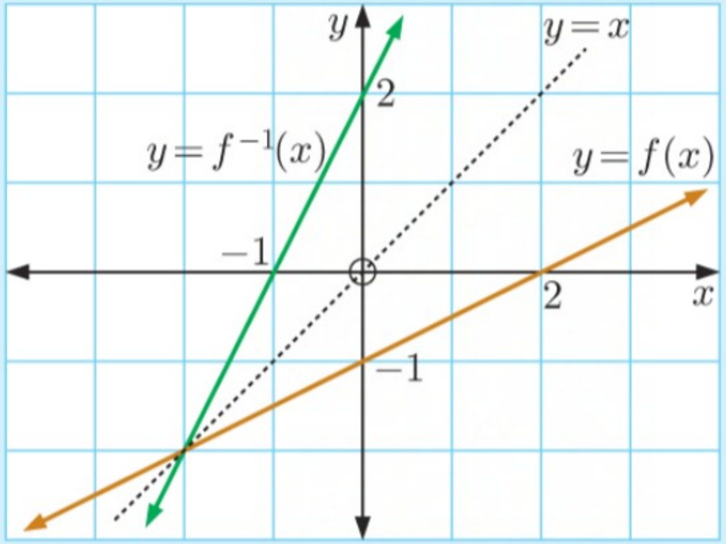
$$\begin{aligned} \therefore\ 2x &= y - 2 \\ \therefore\ y &= 2x + 2 \\ \therefore\ f^{-1}(x) &= 2x + 2 \end{aligned}$$

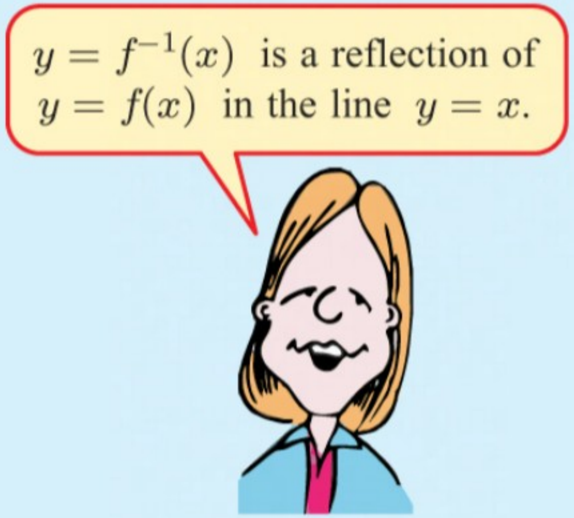
b

$$\begin{aligned} f(f^{-1}(x)) &= f(2x+2) \\ &= \tfrac{1}{2}(2x+2) - 1 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(\tfrac{1}{2}x - 1) \\ &= 2(\tfrac{1}{2}x - 1) + 2 \\ &= x - 2 + 2 \\ &= x \end{aligned}$$

c




$y = f^{-1}(x)$ is a reflection of $y = f(x)$ in the line $y = x$.

1 For each of the following functions:

i Find $f^{-1}(x)$.

ii Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.

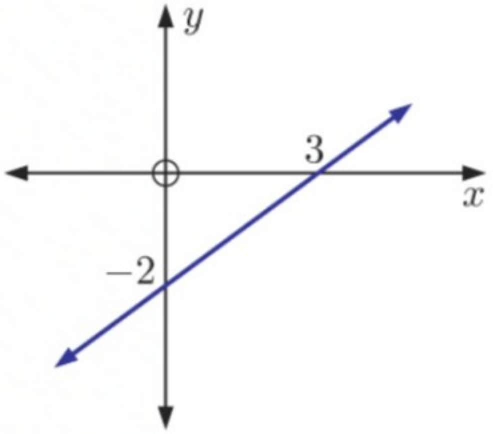
a $f(x) = x + 3$

b $f(x) = 2x + 5$

c $f(x) = \frac{3 - 2x}{4}$

2 Copy the following graphs and draw the graph of each inverse function:

a



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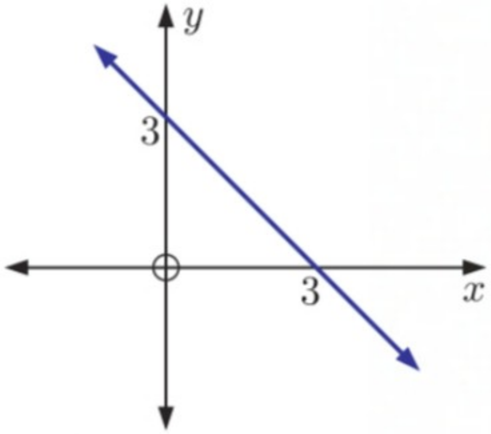

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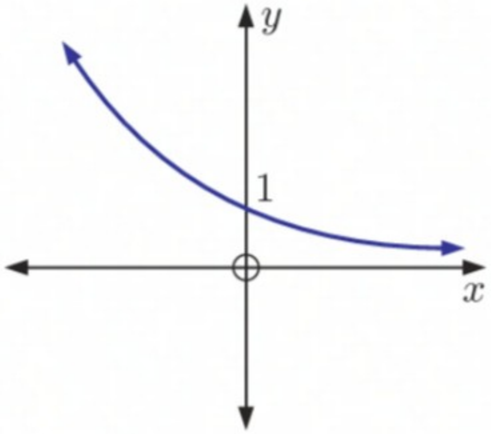


3 Consider $f(x) = 2x + 7$.

a Find $f^{-1}(x)$.

b Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

4 Consider $f(x) = \dfrac{2x+1}{x+3}$.

a Find $f^{-1}(x)$.

b Check that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

5 Consider $g(x) = \dfrac{x}{x-5}$.

a Find $g^{-1}(x)$.

b Check that $g(g^{-1}(x)) = x$ and $g^{-1}(g(x)) = x$.

6 **a** Sketch the graph of $y = x^2$ and reflect it in the line $y = x$.

b Does $f(x) = x^2$ have an inverse function? Explain your answer.

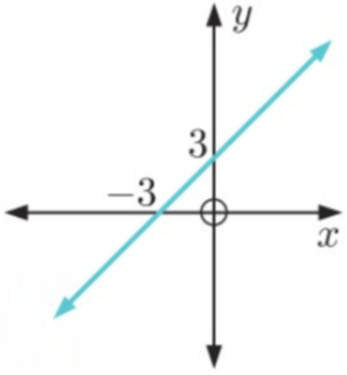
c Does $f(x) = x^2,\ x \geqslant 0$ have an inverse function? Explain your answer.

7 The **horizontal line test** says that *'for a function to have an inverse function, no horizontal line can cut it more than once'*.

a Explain why this is a valid test for the existence of an inverse function.

b Which of the following functions have an inverse function?

i



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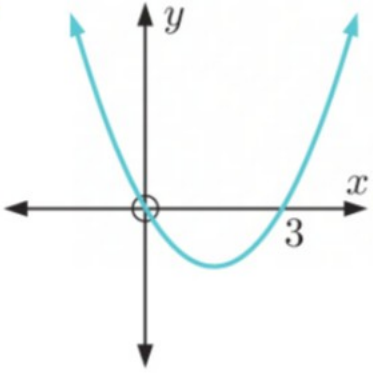

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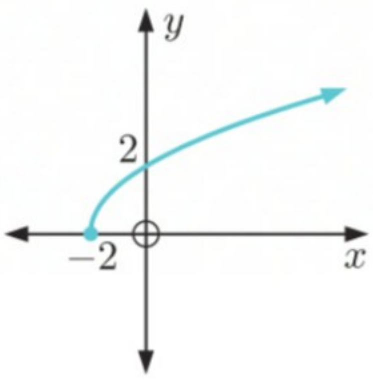

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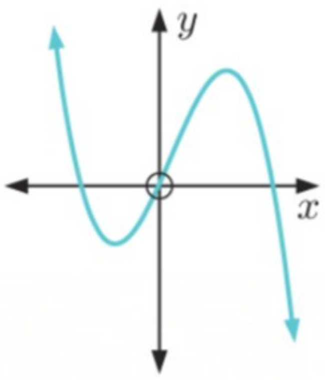


8 **a** Explain why $f(x) = x^2 - 2x + 5$ is a function but does not have an inverse function.

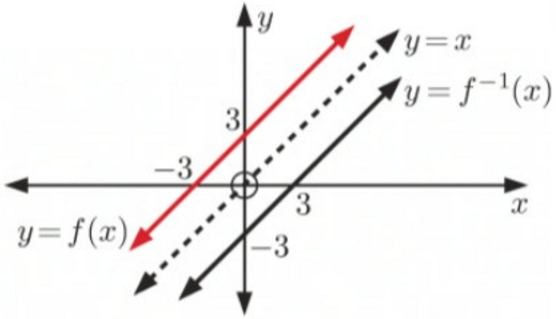
b Explain why $f(x) = x^2 - 2x + 5, \quad x \geqslant 1$ has an inverse function.

c Show that the inverse function of **b** is $f^{-1}(x) = 1 + \sqrt{x - 4}$.

Hint: Swap x and y, then use the quadratic formula to solve for y in terms of x.

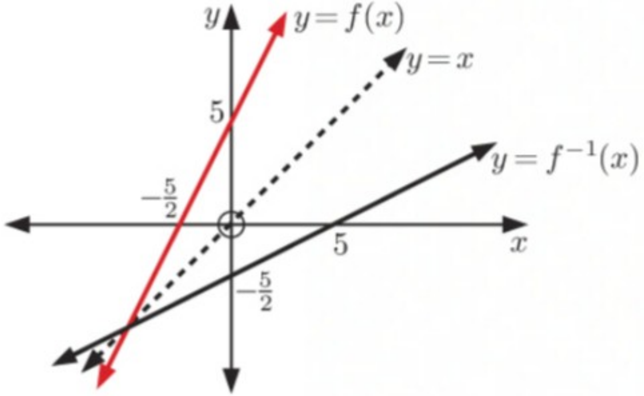
1 **a** **i** $f^{-1}(x) = x - 3$

ii



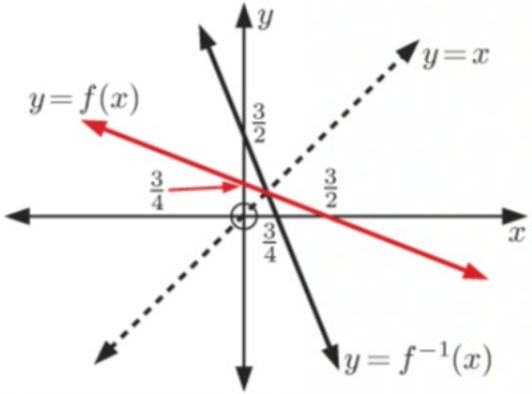
b **i** $f^{-1}(x) = \dfrac{x-5}{2}$

ii

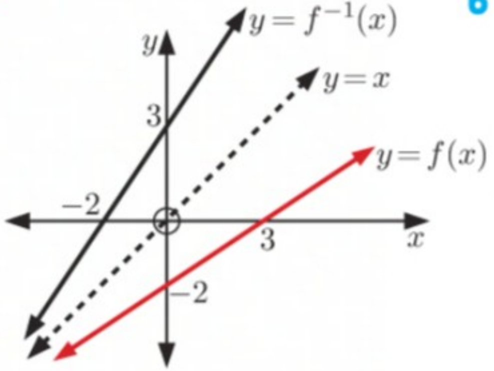


c **i** $f^{-1}(x) = -2x + \frac{3}{2}$

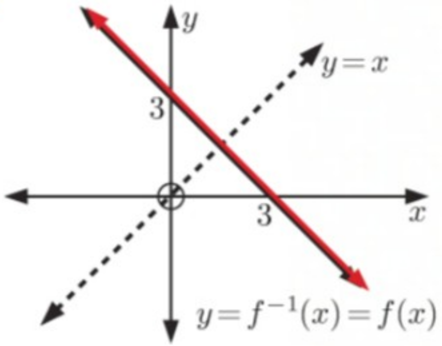
ii



2 **a**



b



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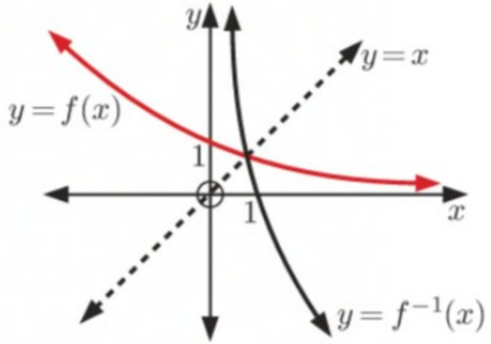


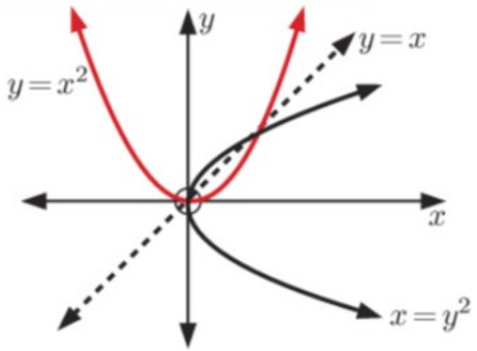
3 **a** $f^{-1}(x) = \dfrac{x-7}{2}$

4 **a** $f^{-1}(x) = \dfrac{1-3x}{x-2}$

5 **a** $g^{-1}(x) = \dfrac{5x}{x-1}$

c $g(6) = 6$, $g^{-1}(1)$ is undefined

6 **a**



b No, as the vertical line test fails.

c Yes, it is $y = \sqrt{x}$ (not $y = \pm\sqrt{x}$).

7 **b** **i** and **iii** have inverse functions.

8 **a** Is a function as it passes the vertical line test, but does not have an inverse as it fails the horizontal line test.

b It passes both the vertical line and horizontal line tests.